

## Gauge Invariance in Electromagnetism

In quantum mechanics, we talk about electromagnetism in terms of  $A_p(x)$  instead of  $\vec{E}$  and  $\vec{B}$ .

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

$$\vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t} = \cancel{-\frac{\partial}{\partial p} A^p} - \frac{\partial}{\partial p} A^p$$

Gauge invariance in electromagnetism refers to the fact that for some given physical fields  $\vec{E}$  and  $\vec{B}$ , ~~these~~ the  $A_p(x)$  fields are not uniquely determined. So we have some freedom to change, or "transform", the  $A_p(x)$  fields and still get the same physical  $\vec{E}$  and  $\vec{B}$  field. Namely, if we let  $\chi$  be any arbitrary function

~~if  $\vec{A}'$  and  $V'$~~

$$\left\{ \begin{array}{l} V \rightarrow V' = V - \frac{\partial \chi}{\partial t} \\ \vec{A} \rightarrow \vec{A}' = \vec{A} + \vec{\nabla} \chi \end{array} \right.$$

produces no change in  $\vec{E}$  or  $\vec{B}$ .

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

$$= \vec{\nabla} \times (\vec{A} + \vec{\nabla} \chi)$$

$$= \vec{\nabla} \times \vec{A} + \vec{\nabla} \times \vec{\nabla} \chi$$

$$= \vec{\nabla} \times \vec{A}$$

$$\vec{E} = -\vec{\nabla} V' - \frac{\partial \vec{A}'}{\partial t}$$

$$= -\vec{\nabla} \left( V - \frac{\partial \chi}{\partial t} \right) - \frac{\partial}{\partial t} (\vec{A} + \vec{\nabla} \chi)$$

$$= -\vec{\nabla} V + \vec{\nabla} \frac{\partial \chi}{\partial t} - \frac{\partial}{\partial t} \vec{A} - \frac{\partial}{\partial t} \vec{\nabla} \chi$$

$$= -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t}$$

Letting  $A^\mu = (V, \vec{A})$ , the gauge transformation is written more compactly as

$$A^\mu \rightarrow A'^\mu = A^\mu - \partial^\mu \chi$$

Next, let  $\psi(\vec{x}, t)$  be some quantum mechanical wavefunction. Let's see what happens when we demand  $\psi$  to be invariant under a spacetime-dependent phase transformation.

$$\psi(\vec{x}, t) \rightarrow \psi'(\vec{x}, t) = e^{i\chi(\vec{x}, t)} \psi(\vec{x}, t)$$

We must immediately notice that the free-particle Schrödinger equation, as well as the relativistic wave equations, are not invariant to this local phase transformation. The freedom to alter the phase of a charged particle's wavefunction locally is only possible if some kind of force field is introduced in which the charged particle moves.

The Schrödinger Equation is modified by

$$\frac{1}{2m}(-i\vec{\nabla})^2 \psi = i \frac{d\psi}{dt}$$



$$\frac{1}{2m}(-i\vec{\nabla} - q\vec{A})^2 \psi = \left(i \frac{d}{dt} - qV\right) \psi$$

So it is precisely the phase transformation of fields from the previous page that allows local phase invariance

$$\begin{cases} \vec{A} \rightarrow \vec{A}' = \vec{A} + \frac{1}{q} \vec{\nabla} \alpha \\ V \rightarrow V' = V - \frac{1}{q} \frac{\partial \alpha}{\partial t} \end{cases}$$